DIFFERENCE IN ESTIMATED LAYER MODULI
FROM STATIC AND DYNAMIC BACK-CALCULATIONS

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ABSTRACT: A falling weight deflectometer (FWD) is commonly utilized to make structural evaluation of pavement systems, which applies impulsive force at a surface of pavement and measure surface deflections. Static back-calculation employs only peak force and peak deflections, while dynamic back calculation uses time histories of both.
Since dynamic analysis of elastic multi-layered systems required in its back-calculation routine of the latter method is extremely time consuming, reducing computational time required for the dynamic analysis is very important. Dynamic reduction analysis method based on Ritz vectors is implemented in the algorithm. In both back-calculation routines, truncated singular value decomposition technique is introduced to improve numerical stability. The results from both methods are presented and discussed.

Keywords: dynamic back-calculation, static back-calculation, truncated singular value decomposition, FWD

1. INTRODUCTION

Falling weight deflectometer (called FWD hereafter) has become a de facto standard nondestructive testing device for pavement and more than three hundred FWDs are now in operation over the world. FWD applies an impulsive force at pavement surface and measures surface deflections at several locations including a point of loading.

There are three methods for structural evaluation from FWD data. The first and simplest approach utilizes only peak surface deflections at a few selected points without back-calculation. The second is static back-calculation which estimates layer moduli from peak values of both force and deflection data by an iterative method. The solution may not be unique. The third is dynamic back-calculation which determines layer moduli so as to match computed deflection histories with measured ones (1)-(4).

The first approach is empirical, often used in practice but does not have a sound theoretical basis. The second is the method to obtain layer moduli which match both measured and computed peak surface deflections, considering the measured peak deflections as pseudo-static deflections. This method is most commonly used but it has been known that the method often leads to erroneous results. It seems that this discrepancy is greatly caused by the difference between the actual phenomenon and the analysis model in addition to numerical instability.

A back-calculation method is composed of forward and backward analyses. Axi-symmetric analysis of multi-layered elastic systems called AAMES (5) is implemented in static back-calculation and FEM is employed in dynamic back-calculation. Back-calculation is in general unstable and requires some sort of regularization (6) and (7). We herein use the Gauss-Newton method with truncated singular value decomposition for both static and dynamic back-calculation. Assuming the initial values of parameters to be identified, the method iteratively updates their values until convergence is achieved.

This paper presents the results of back-calculation from the both methods and gives some discussions.

2. BACK-CALCULATION

2.1 *Gauss-Newton method with regularization*

The solution to this problem requires finding parameter values which measured responses match computed responses. This is so called a non-linear least square problem. Let $\mathbf{u}(\mathbf{X})$ and $\mathbf{u}^*$ be $N \times 1$ vectors of computed and measured responses. Then, the non-linear least square functional can be written as

$$J = \frac{1}{2}\left\{\mathbf{u}^* - \mathbf{u}(\mathbf{X})\right\}^T \left\{\mathbf{u}^* - \mathbf{u}(\mathbf{X})\right\} \tag{1}$$

in which $\mathbf{X}$ is a $M \times 1$ vector of unknown parameters. The value of $\mathbf{X}$ which minimizes equation (1) is the one we want to find. Since this is a non-linear minimization problem, an iterative approach needs to be employed. Using Gauss-Newton method, iterative algorithm can be derived as

$$\mathbf{A}\, d\mathbf{X} = \mathbf{b} \tag{2}$$

where

$$\mathbf{A} = \left[\sum_{i=1}^{N} \frac{\partial u_i}{\partial X_j} \frac{\partial u_i}{\partial X_k}\right] \tag{3}$$

$$\mathbf{b} = \left\{\sum_{i=1}^{N} \left(u_i^* - u_i\right) \frac{\partial u_i}{\partial X_j}\right\} \tag{4}$$

$$d\mathbf{X} = \left\{dX_j\right\} \tag{5}$$

$\mathbf{A}$ is a $M \times M$ matrix, and $\mathbf{b}$ and $d\mathbf{X}$ are $M \times 1$ vectors. Equation (2) is called a normal equation. Assuming an initial value of $\mathbf{X}$, $\mathbf{A}$ and $\mathbf{b}$ can be computed. Although equation (2) is a system of linear equations, it has to be solved with care because of its unstable nature. In order to cope this instability, singular value decomposition is utilized. Considering a symmetry of $\mathbf{A}$, $\mathbf{A}$ can be decomposed and written as

$$\mathbf{A} = \mathbf{V}^T \mathbf{D} \mathbf{V} \tag{6}$$

where $\mathbf{D}$ is a diagonal matrix composed of diagonal elements called singular values and

$$\mathbf{V}^T\mathbf{V} = \mathbf{V}\mathbf{V}^T = \mathbf{I}. \tag{7}$$

Then, the solution $d\mathbf{X}$ can be written as

$$d\mathbf{X} = \mathbf{V}^T\mathbf{D}^{-1}\mathbf{V}\mathbf{b}. \tag{8}$$

If a diagonal element d_{ii} is smaller than a threshold value, $1/d_{ii}$ is taken as zero in the computation of $d\mathbf{X}$. Equations (3) and (4) require the sensitivity computation of response vector. In case of static back-calculation, the surface deflections are computed by AAMES and the sensitivity of the deflections by the same software with a finite difference method. In case of dynamic back-calculation, the response becomes time dependent vector, Equations (3) and (4) have to be integrated over a time domain, which can be numerically carried out. The dynamic and sensitivity analyses required in the computation of Equations (3) and (4) are made by using a finite element method.

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2.2 *Efficient dynamic analysis*

A mathematical model for dynamic analysis of pavement due to impulsive force is formulated based on a finite element model with 20 node isoparametric elements. It usually results in large degrees of freedom and computational time becomes enormous particularly when the system of equations has to be repeatedly solved. To improve computational efficiency, Ritz vector reduction method proposed by Wilson *et al*. is employed. Using Ritz matrix composed of Ritz vectors (6), the reduced system of equation of motion can be written as

$$\mathbf{M}^*\ddot{\mathbf{z}}(t) + \mathbf{C}^*\dot{\mathbf{z}}(t) + \mathbf{K}^*\mathbf{z}(t) = \mathbf{f}^*g(t) \tag{9}$$

$$\mathbf{z}(0) = \mathbf{0}, \quad \dot{\mathbf{z}}(0) = \mathbf{0}$$

where $\mathbf{z}(t) = [z_1(t), z_2(t), ..., z_L(t)]^T$ is reduced response vector, and $\dot{\mathbf{z}}(t)$ and $\ddot{\mathbf{z}}(t)$ in Equation (9) are the velocity and acceleration vectors of the reduced system. $\mathbf{M}^*, \mathbf{C}^*$ and $\mathbf{K}^*$ are the reduced mass, damping and stiffness matrices and their size is $L \times L$, and $\mathbf{f}^*$ is reduced loading vector. The reduced mass matrix $\mathbf{M}^*$ can be expressed as a unit matrix, and other reduced matrices $\mathbf{C}^*$ and $\mathbf{K}^*$ are symmetric but not diagonal.

If the damping matrix is non-proportional, $\mathbf{M}^*$, $\mathbf{C}^*$ and $\mathbf{K}^*$ can not be simultaneously transformed into diagonal matrices. Thus, the equation can be rewritten as a system of first order differential equations:

$$\mathbf{A}\dot{\mathbf{y}} + \mathbf{B}\mathbf{y} = \mathbf{f}_0(t) \tag{10}$$

in which

$$\mathbf{A}=\begin{bmatrix}\mathbf{0} & \mathbf{M}^* \\ \mathbf{M}^* & \mathbf{C}^*\end{bmatrix} \qquad \mathbf{B}=\begin{bmatrix}-\mathbf{M}^* & \mathbf{0} \\ \mathbf{0} & \mathrm{K}^*\end{bmatrix} \tag{11}$$

$$\mathbf{y}=\begin{Bmatrix}\dot{\mathbf{z}} \\ \mathbf{z}\end{Bmatrix} \qquad \mathbf{f}_0(t)=\begin{Bmatrix}\mathbf{0} \\ \mathbf{f}^* g(t)\end{Bmatrix} \ . \tag{12}$$

Since $\mathbf{M}^*, \mathbf{C}^*$ and $\mathbf{K}^*$ are $L\times L$ symmetric matrices, $\mathbf{A}$ and $\mathbf{B}$ are $2L\times 2L$ symmetric matrices, and $\dot{\mathbf{y}}, \mathbf{y}$ and $\mathbf{f}_0$ are $2L\times 1$ vectors. Equation (10) can be solved by a complex mode super position method.

The s-th mode solution of Equation (10) can be written as $\mathbf{y}_s(t)=\mathbf{v}_s(t)e^{l_s t}$. Inserting this equation into Equation (10) yields

$$\left(l_s\mathbf{A}+\mathbf{B}\right)\mathbf{v}_s=\mathbf{0} \tag{13}$$

The above is referred to as an eigenvalue problem. Thus, l_s and $\mathbf{v}_s$ $(s=1,2,...,2L)$ are an eigenvalue and eigenvectors, where the subscript s implies s-th mode. These eigenvalues and eigenvectors are complex, and $\mathbf{v}_s$ is the s-th vector composed of displacement and velocity modes. However, their eigenvalues and eigenvectors become conjugate, because $\mathbf{A}$ and $\mathbf{B}$ are real. Due to the orthogonality of eigenvectors $\mathbf{v}_s$, the following relationship must hold between the s-th and the r-th modes,

$$\mathbf{v}_r^T\mathbf{A}\mathbf{v}_s=\begin{cases}A_s & r=s \\ 0 & r\neq s\end{cases} \tag{14}$$

$$\mathbf{v}_r^T\mathbf{B}\mathbf{v}_s=\begin{cases}B_s & r=s \\ 0 & r\neq s\end{cases} \ . \tag{15}$$

If the mode superposition is applied to Equation (10), the solution $\mathbf{y}(t)$ can be written as

$$\mathbf{y}(t)=\sum_{s=1}^{2L}\mathbf{v}_s r_s(t)\,. \tag{16}$$

Substituting Equation (16) into Equation (10), pre-multiplying the resulting equations by $\mathbf{v}_s^T$ and using the orthogonality of eigenvector $\mathbf{v}_s$ from $\mathbf{A}$ and $\mathbf{B}$, the following relationships are obtained:

$$A_s \dot{r}_s(t) + B_s r_s(t) = a_s g(t) \tag{17}$$

$$A_s = \mathbf{v}_s^T \mathbf{A} \mathbf{v}_s \tag{18}$$

$$B_s = \mathbf{v}_s^T \mathbf{B} \mathbf{v}_s \tag{19}$$

$$a_s g(t) = \mathbf{v}_s^T \mathbf{f}_0(t)\,. \tag{20}$$

Similarly, pre-multiplying Equation (13) by $\mathbf{v}_s^T$ and considering the orthogonality,

$$l_s A_s + B_s = 0\,. \tag{21}$$

Then the following first order differential equation can be obtained:

$$\dot{r}_s(t) - l_s r_s(t) = \frac{a_s}{A_s} g(t)\,. \tag{22}$$

If $g(t)$ can be expressed in a piece-wise linear form, it can be written as

$$g(t) = \frac{g(t_{n+1}) - g(t_n)}{Dt}(t - t_n) + g(t_n) \tag{23}$$

between t_n and $t_{n+1} = t_n + \mathrm{D}t$. The solution of equation (22) can be expressed as the following recurrence formula:

$$\begin{aligned} r_s(t_{n+1}) = r_s(t_n) e^{l_s Dt} &+ \frac{a_s}{A_s\, l_s}\left(e^{l_s Dt} + \frac{1}{l_s Dt} - \frac{e^{l_s Dt}}{l_s Dt} \right) g(t_n) \\ &- \frac{a_s}{A_s\, l_s}\left(1 + \frac{1}{l_s Dt} - \frac{e^{l_s Dt}}{l_s Dt} \right) g(t_{n+1})\,. \end{aligned} \tag{24}$$

The solution of Equation (11) at $t = t_{n+1}$, one can write as follows:

$$\mathbf{y}(t_{n+1}) = \sum_{s=1}^{2L} \mathbf{v}_s r_s(t_{n+1}) \,. \tag{25}$$

The above equation gives both the velocity and displacement response in a complex form. Pre-multiplying by Ritz vectors and taking a real part, the surface deflection is easily obtained.

2.3 *Sensitivity analysis*
Back-calculation requires sensitivity of response with respect to unknown parameters. The sensitivity can be computed from the following reduced sensitivity equation:

$$\mathbf{M}^* \frac{\partial \ddot{\mathbf{z}}}{\partial X_j} + \mathbf{C}^* \frac{\partial \dot{\mathbf{z}}}{\partial X_j} + \mathbf{K}^* \frac{\partial \mathbf{z}}{\partial X_j} = \mathbf{h}_j^* \tag{26}$$

in which

$$\mathbf{h}_j^* = -\frac{\partial \mathbf{C}^*}{\partial X_j} \dot{\mathbf{z}} - \frac{\partial \mathbf{K}^*}{\partial X_j} \mathbf{z} \,. \tag{27}$$

From equation (26), initial conditions $(t = 0)$ are:

$$\frac{\partial \mathbf{z}}{\partial X_j} = \frac{\partial \dot{\mathbf{z}}}{\partial X_j} = \mathbf{0} \,. \tag{28}$$

The solution of equation (26) can be written similarly to that of Equation (9). After the sensitivity is computed, the sensitivity of surface deflections with respect to X_j can be obtained by pre-multiplying the Ritz matrix.

3. NUMBER OF DEFLECTION DATA AND BACK-CALCULATED RESULTS

In case of static back-calculation, the number of deflectometers must be greater or equal to the number of unknown parameters. Dynamic back-calculation, however, does not require this condition. Choosing a four layer model as shown in Figure 1, we examined how the number of deflectometers affects back-calculated results. The number of deflectometers and their locations used for back-calculation are described in Table 1. In the dynamic back-calculation , unknown parameters chosen are layer modulus E_i and damping coefficient C_i . The damping matrix is formed by replacing a layer modulus E_i in the stiffness matrix by C_i, which is herein called a layer damping coefficient. They are regarded as independent parameters in this study, the total number of unknowns become eight in case of a four layer system. Initial values of layer moduli are 6000MPa, 500MPa, 200MPa and 60MPa and those of damping coefficients are chosen as 5% of corresponding layer moduli.

Back-calculated results are presented in Table 2. It is found from the table that as many as eight parameters can be identified from deflection measurement at only one location, when dynamic back-calculation is conducted. It implies that the time dependent deflection contains more information than static deflection.

Figure 2 shows the comparison of measured deflections and computed deflections after unknown parameters are identified. In Figure 2(a), computed D60 deflection is sought to match with measured D60 deflection, while other deflections are disregarded when estimating the unknown parameters. Thus, only D60 deflections of calculated and measured among others show good agreement. When three deflections are sought to match as in the case 4 of Table 1, those deflections of computed and measured agree well, but other responses do not show good coincidence. If all measured and computed deflections are used for identification, all responses demonstrate reasonably good agreement. These facts indicate non-uniqueness of the pavement back-calculation problem.

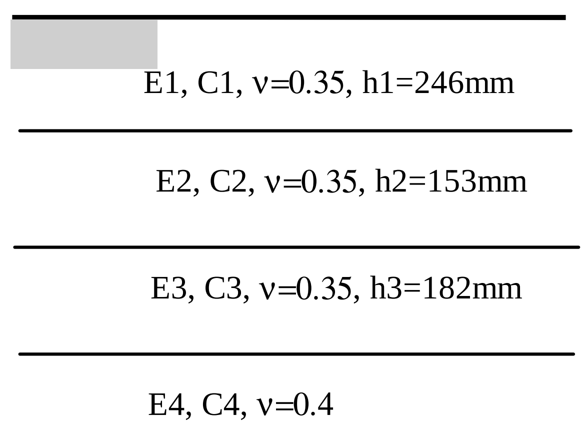


Figure 1 Four layer pavement profile

Table 1 Deflections used for back-calculation

case	Deflections [a]
1	D0
2	D60
3	D90
4	D0, D30, D60
5	D0, D30, D60, D90, D120
6	D0, D30, D45, D60, D90, D120, D150

[a]The number after D indicates the distance in cm from the center of loading plate

Table 2 Back-calculated results[a,b]

Case	No. of Itr.	Layer1	Layer2	Layer3	Layer4	Eq.(1)
1	15	9071 5.18	637 0.384	172 0.116	26 0.040	5.74E-04
2	8	7209 3.85	581 0.288	182 0.091	42 0.035	3.37E-04
3	4	6541 3.23	481 0.239	149 0.076	51 0.030	2.58E-04
4	11	7411 5.12	557 0.378	155 0.117	37 0.045	3.35E-04
5	20	5221 7.06	573 0.515	158 0.161	53 0.019	1.40E-04
6	19	6525 6.62	582 0.492	130 0.156	53 0.022	1.47E-04

[a]Upper number is a layer modulus in MPa

[b]Lower number is a damping coeffcient in Ns/m^2

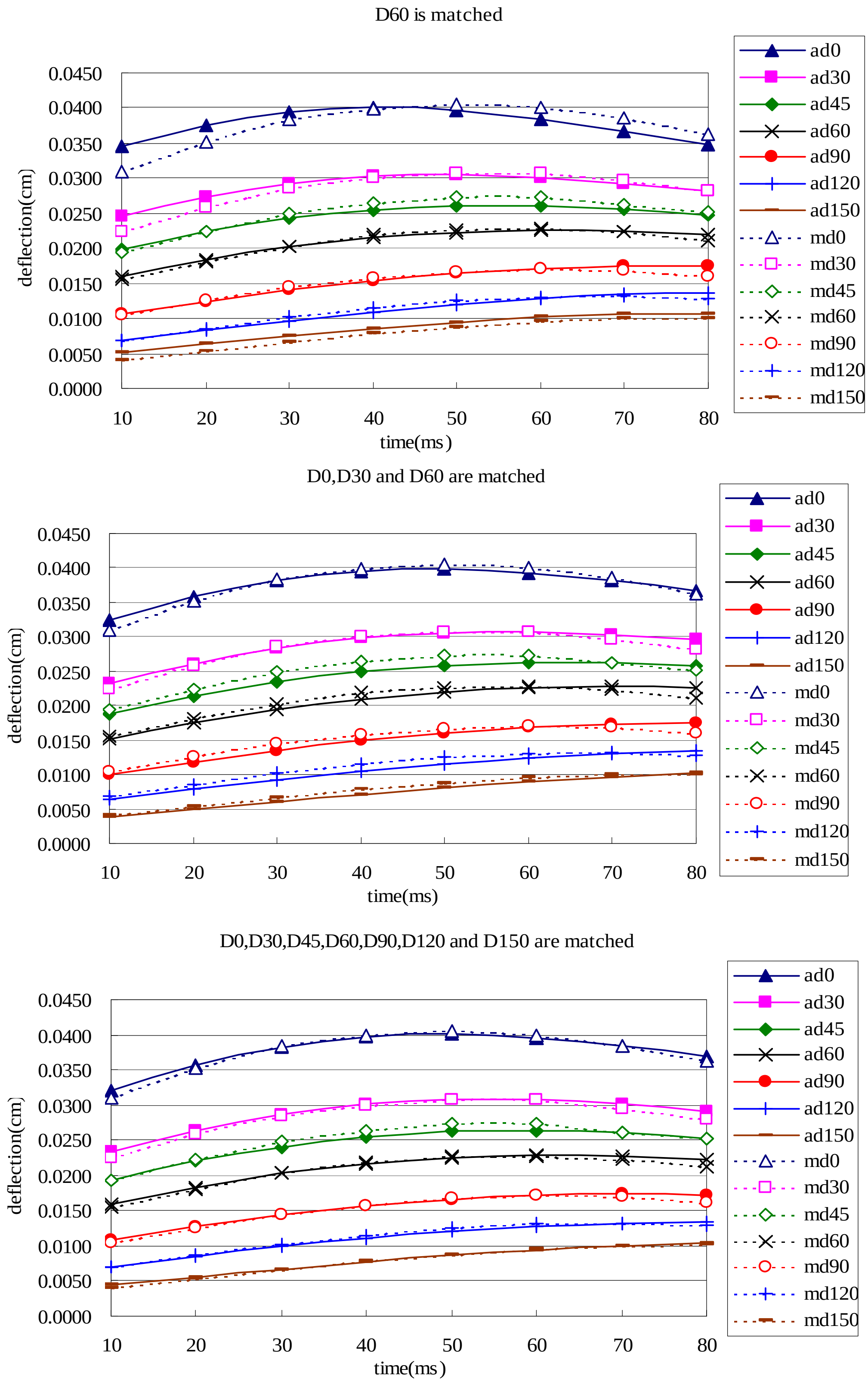


Figure 2 Comparison of measured and computed deflections

4. COMPARISON BETWEEN STATIC AND BACK-CALCULATED RESULTS

Joint FWD test was conducted on March 29-31, 1993 at Japan Public Work Institute. Sixteen FWDs took part in the test at a roller compacted concrete pavement (RCCP). The pavement profile is illustrated in the Figure 3. The base M-30 refers to a mechanically stabilized layer with maximum grain size of 30mm and C-40 means a crusher-run with maximum size of 40mm.

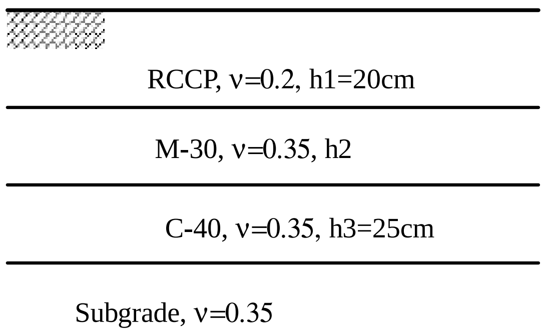


Figure 3 RCCP Pavement Profile

All FWD participated in the test are manufactured by KUAB. Static and dynamic back-calculations are carried out on measured deflection data. In the static back-calculation, we utilized the peak force and peak deflections of the time dependent data. The layer moduli from static back-calculation as well as the layer moduli and the corresponding layer damping coefficients from dynamic back-calculation are illustrated in Figure 4. FWD5 and FWD6 are falling weight deflectometers for airport facility with a loading plate diameter of 45cm. These

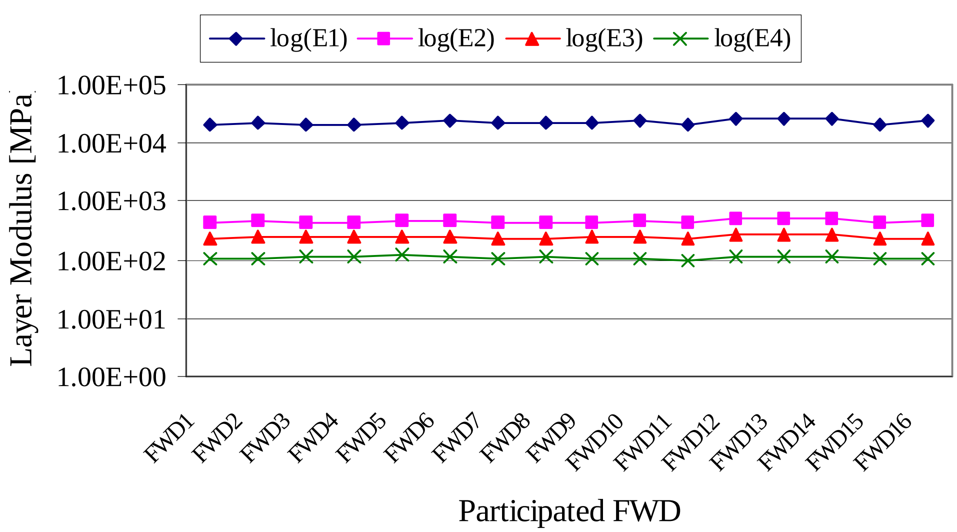
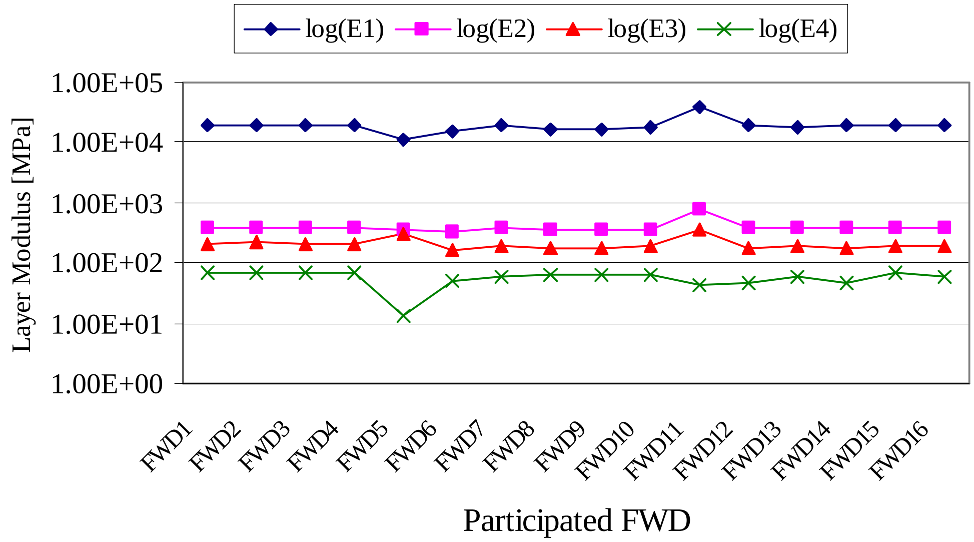
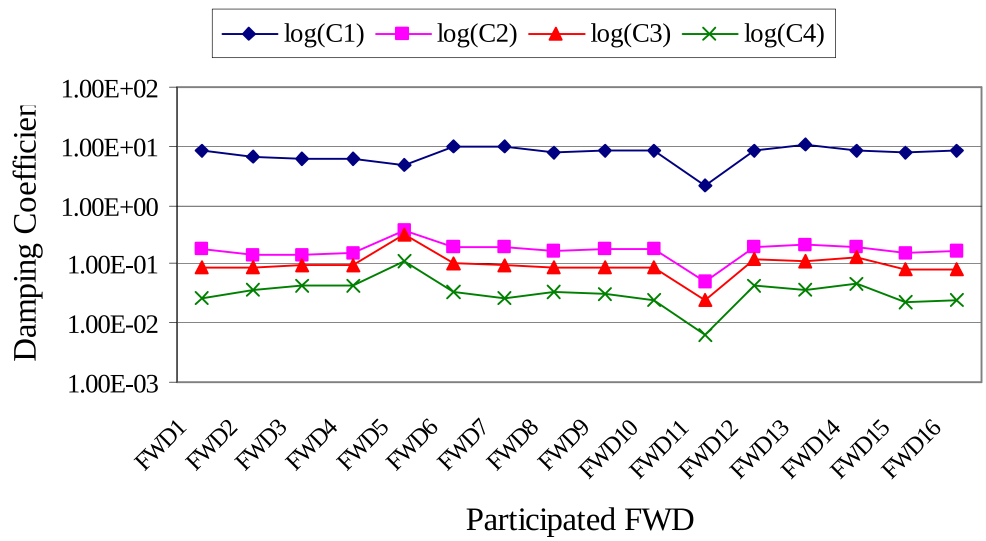


Figure 4 Comparison of Static and Dynamic Back-Calculation Results (a) Static Back-Calculation



(b) Dynamic Back-Calculation (Layer Modulus)



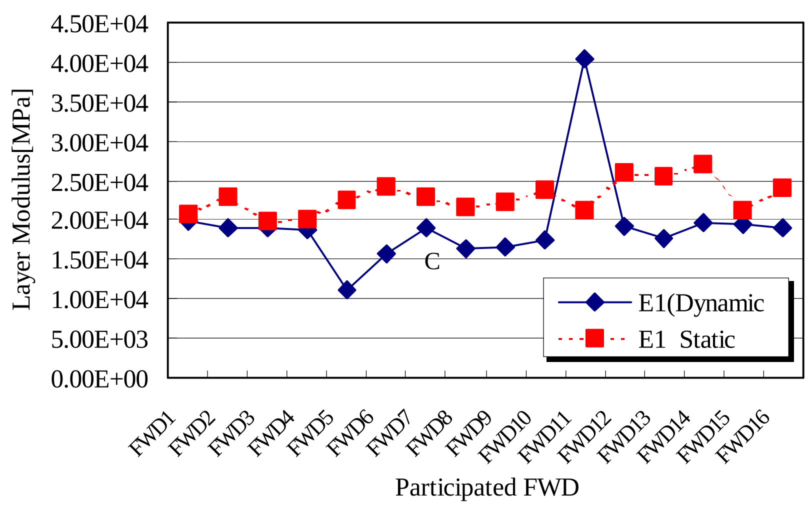
(b) Dynamic Back-Calculation (Layer Damping Coefficient)

Figure 4 (Continued) Comparison of Static and Dynamic Back-Calculation Results

two FWDs applied a peak force of about 176kN to the pavement, while a diameter of the other 14 FWDs is 30cm and the 49kN peak loading is applied during the test. The moduli from static back-calculation are presented in Figure 4(a). Since the vertical axes of Figure 4 are log scales, the back-calculated results appear to be nearly same for each parameter. However in the case of dynamic back-calculation, both layer modulus and the corresponding damping coefficients can be estimated, which are presented in Figure 4(b) and (c). The moduli and the corresponding damping coefficients of FWD5 and FWD11 appear a little different from those of the rests. Estimated concrete modulus of this pavement is about 20,000MPa, which is smaller than the values we normally obtain at concrete pavement by back-calculation.

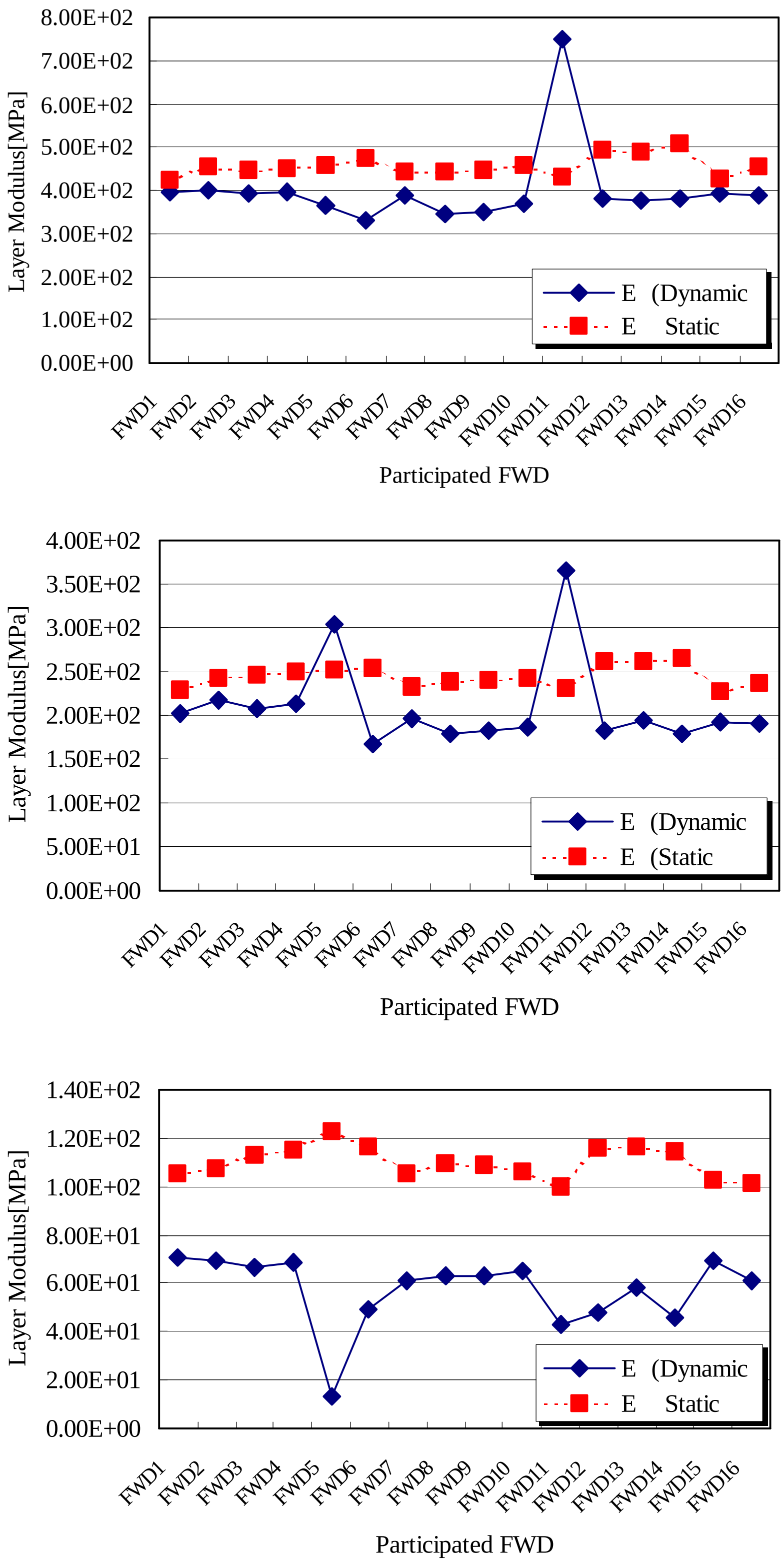
In order to observe the differences in the results, the moduli from static and dynamic back-calculation are compared layer by layer in Figure 5 by using a normal scale. The dynamic back-calculation moduli from FWD5, FWD6 and FWD11 are apparently different. The differences in the moduli for FWD5 and FWD6 are most likely due to higher intensity of loading. The difference in the moduli for FWD11 is caused by erroneous deflection data in which D90 deflection is found smaller than D150 deflection. It is also observed from Figure 5 that although the moduli of concrete and base show small difference between the results from both methods, the subgrade modulus by static back-calculation is much greater than that by dynamic back-calculation. The reason is that because of very short duration of FWD loading, deformation in subgrade due to impulsive loading is smaller than the deformation due to static loading of the same loading intensity. It is also found that when pavement response seems non-linear or measured deflection data contains some error, its effect appears more clearly on the dynamic back-calculation results than the static results.

Figure 6 demonstrates comparison between measured and computed surface deflections after back-calculation. The figure for FWD5 shows that although measured and computed deflections agrees relatively well, little different behavior can be observed between the two. However all response curves for FWD8 are very similar, although matching of D150 deflections may not be good. As for FWD11, all computed responses do not coincide with measured ones, because the peak values of measured D90 and D150deflections are reversed in magnitude.



(a) RCCP Layer

Figure 5 Layer-wise Comparison of Modulus



(b) M-30 Layer

(c) C-40 Layer

(d) Subgrade

Figure 5 Layer-wise Comparison of Modulus (continued)

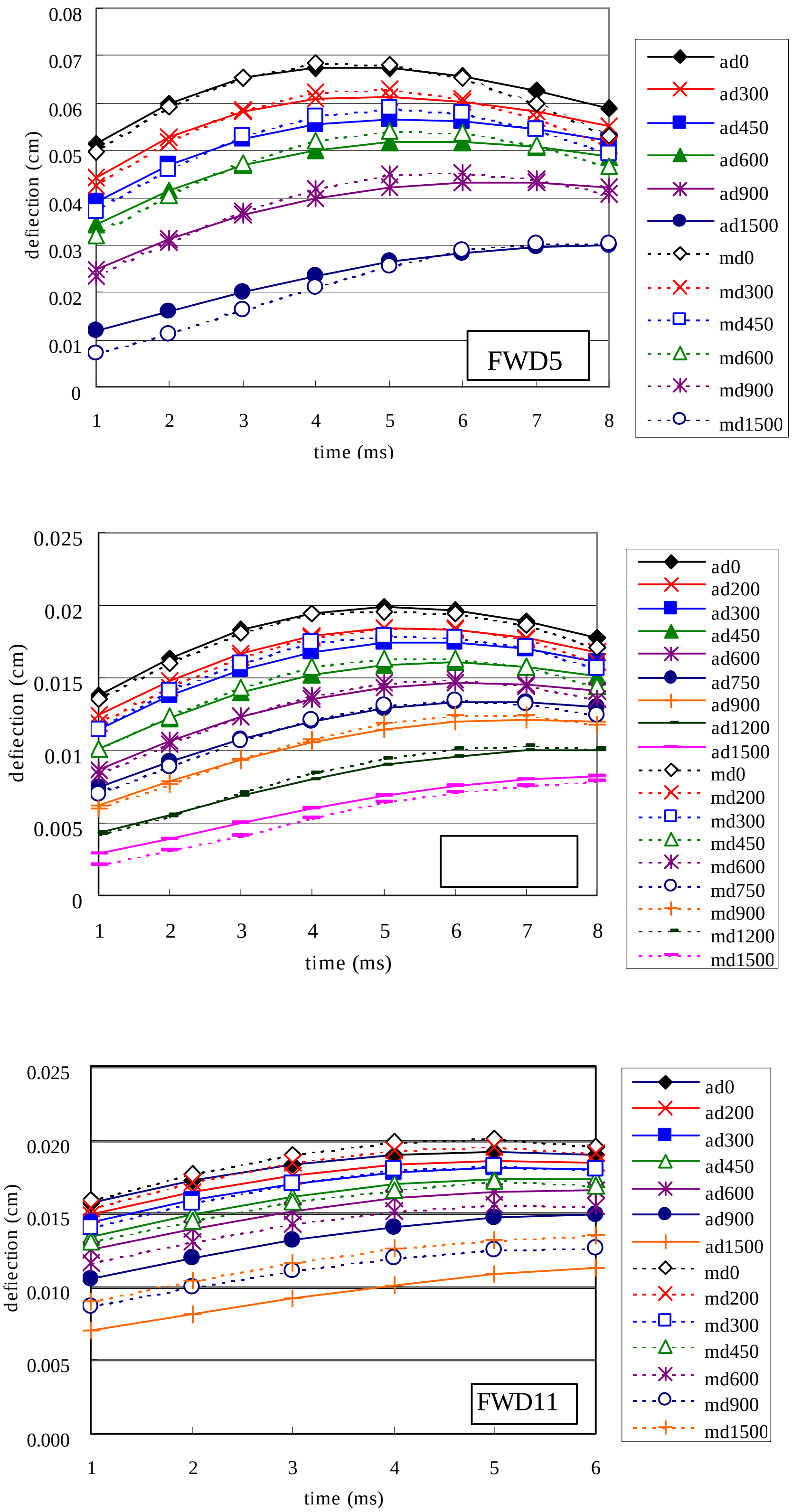


Figure 6 Comparisons between measured and computed deflections

5. CONCLUSIONS

Static and dynamic back-calculation methods are briefly described above and the results obtained by both methods were compared. The following conclusions can be made:

1. Not only layer modulus but also layer damping coefficient can be estimated by dynamic back-calculation
2. More number of unknown parameters can be identified with less number of deflection sensors by employing dynamic back-calculation.
3. Estimates by dynamic back-calculation tend to be smaller than those by static back-calculation. This trend is particularly conspicuous in the subgrade modulus.
4. If dynamic back-calculation is conducted, effects of material non-linearity as well as erroneous deflection data are clearly appeared on back-calculated results.

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References

1) Matsui, K., Nishizawa, T. and Kikuta, Y. Time domain back-calculation of pavement, Structural Materials Technology, SPIE Volume 3400, An NDT Conference, 31 March – 3 April, San Antonio, Texas,
2) Matsui, K., Kikuta, Y., Nishizawa, T. and Kasahara, A. Comparative study of backcalculated results from FWDs with different loading duration., Nondestructive Testing of Pavements and Backcalculation of Moduli: Third Volume (editors: Tayabji, S.D. and Lukanen, E.O.), STP 1375, ASTM, 470-483, 2000.
53) Shoukry, S.N. and William, G.W. Performance evaluation of backcalculation algorithms through three-dimensional finite element modeling of pavement structures, Transportation Research Record 1655, 152-160, 2000.
4) Dong, Q., Matsui, K. and Yamamoto, K. Time domain backcalculation of pavement structure using 3D FEM with Ritz vector method, International Journal of Geomechanics, Number 3, 325-336, 2001
5) Matsui, K., Maina, J.W. and Inoue, T. Axi-symmetric Analysis of Elastic Multi-layer System Considering Interface Slip. 2nd International symposium on maintenance and rehabilitation of pavements and technology control, Alabama, USA, 2001, CD-ROM.
6) Lawson, C.L. and Hanson, R.J., Solving Least Square Problems, Prentice Hall, Inc., 1974.
7) Hansen, P.C. and O'Leary, D.P., The Use of the L-curve in the Regularization of Discrete Ill-posed Problems, SAIAM Journal on Scientific Computing, Vol. 14, No.6, pp.1487-1503., November, 1993.
8) Wilson E.L., Yuan, M.W. and Dicken, J.W. Dynamic analysis by direct superposition of Ritz vectors, Earthquake Eng. Struct. Dyn., 10, 813-821, 1982.